Contagion! The Spread of Systemic Risk in Financial Networks

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The Goal of the Book

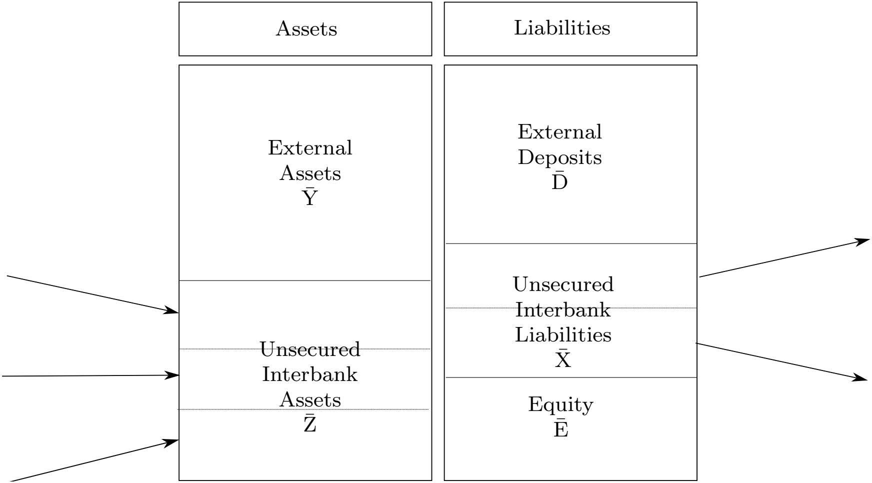
Main Aim

To crystallize a stream of systemic risk research that focuses on the basic modelling structure and ensures some kind of mathematical tractability. At the same time, it should be scalable in complexity to allow a great deal of reality in actual finance network specifications.

Basic Modelling Strategy

1. Propose a definition of Random Financial Network of N banks, that gives a minimal description of the system at any time.
2. Propose a Cascade Mechanism.
3. Hit the system with a random shock at time 0 and determine the evolution of the resultant cascade.
4. Compare to exact $N \to \infty$ analytics that arise by "extending the logic of percolation theory".

Eisenberg-Noe 2001 Model: Balance Sheets



EN2001 Assumptions

1. External debt $\bar{\mathrm{D}}$ is senior to interbank debt $\bar{\mathrm{X}}$;
2. All interbank debt is of equal seniority;
3. There are no losses due to bankruptcy charges.

Also:

1. Exposure $\bar{\Omega}_{vw}$: what v owes w.
2. Equity and Default buffers: $E_v = \max(0, \bar{\Delta}_v)$.
3. Limited Liability: A bank is defaulted at cascade step n if and only if $\Delta_v^{(n)} \leq 0$.

EN2001 Default Buffer Mapping

1. If $\Delta_w^{(n)}$ denotes the default buffer after n cascade steps, then

$$\Delta_w^{(n)} = \Delta_w^{(0)} - \sum_v \bar{\Omega}_{vw} \left(1 - h(\Delta_v^{(n-1)}/\bar{X}_v)\right)$$

2. Threshold function

$$h(x) \quad = \max(x+1,0) - \max(x,0)$$

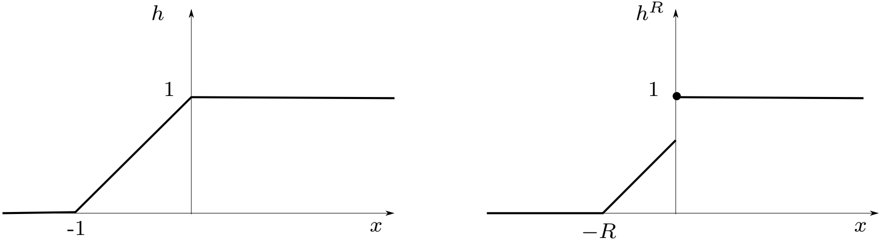
determines fractional recovered value of defaulted interbank assets.

3. As $n \to \infty$, buffers $\Delta_w^{(n)}$ converge to unique fixed point $\Delta^+ = \{\Delta_v^+\}$ of solvency cascade mapping.

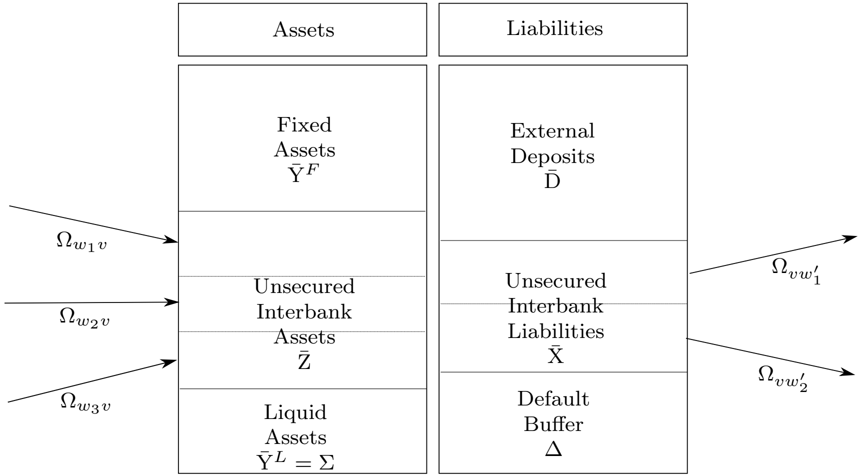
Gai-Kapadia 2010 Default Model

1. GK assumes zero recovery at default and is formally identical to EN2001, but with threshold function h replaced by $\tilde{h}(x) = \mathbf{1}_{x>0}$.
2. Partial recovery at default can be modelled by

$$h^R(x) = Rh(x/R) + (1-R)\tilde{h}(x)\ .$$



Illiquidity Cascades: Balance Sheets



Illiquidity Cascade: Gai-Kapadia 2010b

1. At time 0, some banks experience deposit withdrawals that deplete their liquidity buffer $\Sigma_v := \mathrm{Y}_v^L$ (allowing it to go negative).
2. Bank v with $\Sigma_v \le 0$ reacts by hoarding liquidity; its debtor banks $w \in \mathcal{N}_v^+$ each receive a liquidity shock.
3. Under 100% hoarding, cascade mapping at step n is

$$\Sigma_v^{(n)} = \Sigma_v^{(0)} - \sum_{w \in \mathcal{N}_v^+} \bar{\Omega}_{vw} (1 - \tilde{h}(\Sigma_w^{(n-1)}/\bar{\mathrm{Z}}_w))$$

4. Formally identical to GK 2010 Default Cascade under interchange of assets and liabilities.

Illiquidity Cascade 2: Seung Hwan Lee 2013 Model

1. At time 0, banks experience deposit withdrawals $\Delta d_v \geq 0$.
2. These are paid immediately first by liquid assets $\bar{\mathrm{Z}} + \bar{\mathrm{Y}}^L$, then fixed assets $\bar{\mathrm{Y}}^F$.
3. Debtor banks receive liquidity shocks;
4. Each bank v has initial liquidity buffer $\Sigma_v^{(0)} = -\Delta d_v \leq 0$
5. After $n-1$ cascade steps, then

$$\Sigma_w^{(n)} = \Sigma_w^{(0)} - \sum_v \bar{\Omega}_{wv} \left(1 - h(\Sigma_v^{(n-1)}/\bar{\mathrm{Z}}_v)\right)$$

6. Mathematically identical to a restricted version of EN 2001!

Asset Fire Sale Cascades

(c.f. Cifuentes et al 2005 and Caccioli et al 2012)

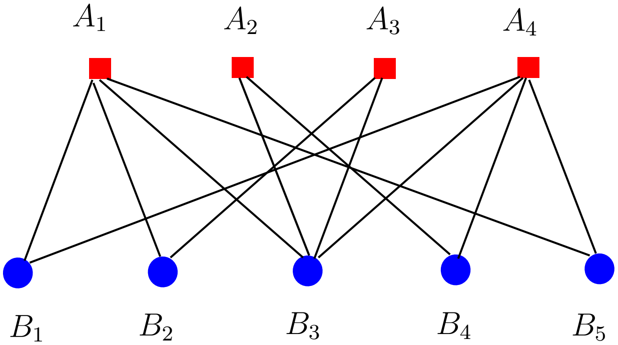


Figure: A bipartite graph with 5 banks (blue nodes) co-owning 4 assets (red nodes).

Asset Fire Sales

Banks $v \in \mathcal{N} = \{1, 2, \ldots, N\}$, Assets $a \in \mathcal{M} = \{1, 2, \ldots, M\}$.
$\bar{s}_{av}$ = amount of asset a held by bank v.
On the nth cascade step:

1. When default buffer $\Delta_v^{(n)}$ hits a threshold, v begins to liquidate assets.
2. Amount $s_{av}^{(n)}$ of asset a held by bank v after n cascade steps is determined by $\Delta_v^{(n)}$.
3. The new mark-to-market price is determined by the total amount sold through an inverse demand function

$$p_a^{(n+1)} = d_a^{-1}\Big(\sum_v (\bar{s}_{av} - s_{av}^{(n)})\Big)$$

4. Banks mark-to-market to compute their new buffers $\Delta_v^{(n+1)}$.

Asset Fire Sale Cascades

1. Complex cascades result even with no interbank sector $\bar{\Omega} = 0$.
2. Each blue node v is governed by a buffer variable $\Delta_v^{(n)}$
3. Each red node a is governed its price $p_a^{(n)}$, which can be considered as a buffer variable.
4. One buffer per node!
5. Global cascades can start either in banks or in assets: once it starts it doesn't matter much where it started.

Single Buffer Models

1. In all these models, each node's behaviour, and hence the cascade itself, is determined by a single buffer Δ_v, Σ_v or p_a.
2. Single buffer models can easily account for multiple thresholds of behaviour.

A Double Buffer Model

1. In more complex models, banks' behaviour is determined by two or more buffers.
2. HCMS 2014 introduces a double cascade model of illiquidity and insolvency, intertwining two buffers Δ_v, Σ_v, that combines the essence of both [GK, 2010a] default cascade and [GK, 2010b] liquidity cascade.

Question

What effect does a bank's behavioural response to liquidity stress have on the probable level of eventual defaults in entire system?

Essential Modelling Assumptions

To have analytical results, not just Monte Carlo simulation experiments, should assume locally tree-like independence properties (LTI):

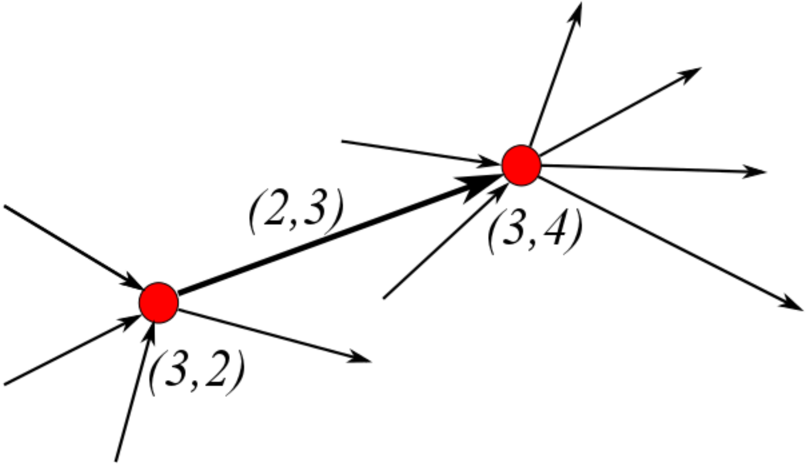
1. Network of bank counterparties is large, sparse and not too heterogeneous.
2. Interbank exposures and bank characteristics are families of random variables with hierarchical dependence.
3. Cascade mechanisms that model bank behaviour are "compatible" with network structure.

Definition: Random Financial Networks

Three layers of mathematical structure:

1. Skeleton: random (directed) graph $(\mathcal{N}, \mathcal{E})$ with banks $v \in \mathcal{N}$ and edges or links $\ell = (vw) \in \mathcal{E}$ represent a non-negligible counterparty relation (or directed interbank exposure).
2. Conditioned on skeleton $(\mathcal{N}, \mathcal{E})$: Random buffers $\bar{\Delta}_v$ for each bank.
3. Finally, conditioned on skeleton and balance sheets: Random exposures $\bar{\Omega}_\ell$ for each link $\ell = (w, v) \in \mathcal{E}$.

Directed Graphs: 2 Nodes and 1 Edge



Directed Assortative Configuration Model

1. Skeleton $(\mathcal{N}, \mathcal{E})$ is a directed configuration (LT) random graph with specified node and edge degree type distribution matrices $\{P_{jk}, Q_{kj}\}$.
2. $P_{jk} = \mathbb{P}[v \in \mathcal{N}_{jk}]$ where $\mathcal{N}_{jk}$ is the set of nodes with $\deg^-(v) = j, \deg^+(v) = k$. Marginals: $P^+_k = \sum_j P_{jk}, P^-_j = \sum_k P_{jk}$.
3. $Q_{kj} = \mathbb{P}[\ell \in \mathcal{E}_{kj}]$, marginals $Q^+_k = \sum_j Q_{kj}, Q^-_j = \sum_k Q_{kj}$.
4. When bivariate distribution $Q \neq Q^+Q^-$ the network is called assortative.
5. P and Q must be consistent:

$$\begin{aligned} z &:= \sum_{jk} jP_{jk} = \sum_{jk} kP_{jk} \\ Q^+_k &= kP^+_k/z, \quad Q^-_j = jP^-_j/z \ . \end{aligned}$$

Assortative Configuration Graph (ACG) Theorem

Theorem

1. *Conditioned on node-degree sequence* $X = (j_i, k_i), i \in [N]$ *for finite* N*, wiring sequence* $W = \{\ell_t = (v_t^+, v_t^-), t \in [L]\}$*:*

$$\mathbb{P}[W|X] = C^{-1} \prod_{kj} (Q_{kj})^{|\{t:\ell_t \in \mathcal{E}_{kj}\}|}$$

2. *For any* *configuration* $\tilde{g}$ *with* M *nodes* $v_i \in \mathcal{N}_{j_i k_i}$ *and* $L \geq M+1$ *edges*

$$\lim_{N\to\infty} N^{M+1-L} \mathbb{P}[v_i \in \mathcal{N}_{j_i k_i}, i \in [M], \tilde{g}] = Z^{-1} \prod_{m\in[M]} \left(\frac{P_{j_m k_m} j_m! k_m!}{(j_m - \tilde{j}_m)!(k_m - \tilde{k}_m)!} \right) \prod_{\ell\in[L]} \frac{Q_{k_\ell j_\ell}}{Q^+_{k_\ell} Q^-_{j_\ell}}$$

Other Potential Random Graph Constructions

1. Preferential attachment models;
2. Preferential attachment and detachment models;
3. Inhomogeneous Random Graphs: each bank $v \in \mathcal{N} = \{1, \ldots, N\}$ is independently assigned a random type $t_v \sim F$ with values in type space $\mathcal{T}$; Conditioned on $\{t_v\}$, edges $(w, v) \in \mathcal{E}$ are drawn independently with probability

$$\frac{K(u(t_v), u(t_w))}{N - 1 + K(u(t_v), u(t_w))}$$

where $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is symmetric and $u : \mathcal{T} \to \mathbb{R}_+$.

Percolation Logic

1. A large sparse random graph typically has few short closed cycles: call it locally tree-like.
2. Implies all sums over counterparties are almost (conditionally) independent: local tree-like independence (LTI).
3. Vulnerable edge: a directed edge $(w, v) \in \mathcal{E}$ such that $\bar{\Omega}_{wv} \geq \bar{\Delta}_v$.
4. Percolation Logic: large scale cascades starting from a single seed default are only possible if the directed subgraph $\mathcal{E}_V$ of vulnerable edges has a large strongly connected component Giant-SCC.
5. Because of the LTI property, size $|\mathrm{GSCC}|$ can be analyzed exactly in the limit $N \to \infty$.

Zoology of Components of Directed Graphs

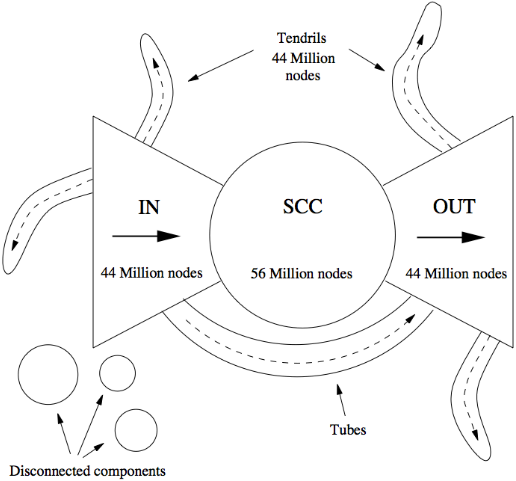


Figure: The connected components of the World Wide Web in 1999. (Source: Broder et al 2000.)

Percolation Theory on Random Graphs $\mathcal{G}$

Question

What is the size of the largest connected cluster $\mathcal{C}$ in $\mathcal{G} = (\mathcal{N}, \mathcal{E})$?

Theorem (Molloy-Reed 2000)

Let $\mathcal{G}^{(N)}, N := |\mathcal{N}| \to \infty$ be a "well-behaved" random graph sequence with the locally tree-like (LT) property. Then

$$\mathbb{E}[|\mathcal{C}|] \overset{N\to\infty}{=} N(1 - g(\xi^*))[1 + o(1)]$$

Here $\xi^* = \frac{g'(\xi^*)}{g'(1)}$ and $g(x) := \sum_k \mathbb{P}[k_v = k]x^k$ is the generating function of the asymptotic degree distribution.

Both Configuration Graphs and Inhomogeneous Random Graphs have the LT property.

Percolation Condition

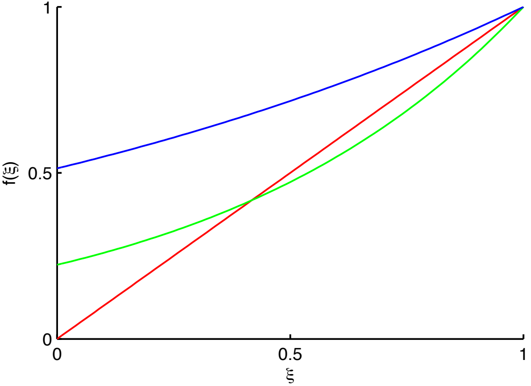


Figure: Supercritical case is found for the green curve, which has a non-trivial fixed point $\xi^* < 1$. The blue curve has only the trivial fixed point at $\xi^* = 1$, and corresponds to a sub-critical random graph.

Cascade Dynamics = Bootstrap Percolation?

1. Bootstrap Percolation is a dynamic version of percolation introduced in 1979 by Chalupa, Leath and Reich for magnetic systems on regular lattices.
2. It follows the growth of connected clusters of nodes $v \in \mathcal{N}$ that become “activated” when the number of its active neighbours exceeds a threshold.
3. Exact analytic asymptotics are sometimes possible on LTI networks.
4. Watts’ 2002 Information Cascade Model is a basic example of Bootstrap Percolation.

Extended Gai-Kapadia RFN

1. Skeleton graph is a directed assortative configuration random graph with $\{P_{jk}, Q_{kj}\}$.
2. Conditionally on the skeleton, buffers $\bar{\Delta}_v$ are a collection of independent non-negative random variables with

$$\mathbb{P}[\bar{\Delta}_v \leq x | v \in \mathcal{N}_{jk}] = D_{jk}(x), \; x \geq 0 \; . \qquad (1)$$

3. Conditionally on the skeleton, exposures $\bar{\Omega}_\ell$ form a collection of independent positive random variables, independent as well from the default buffers $\bar{\Delta}_v$ with

$$\begin{aligned} W_{kj}(x) &= \mathbb{P}[\bar{\Omega}_\ell \leq x | \ell \in \mathcal{E}_{kj}], \\ w_{kj}(x) &= W'_{kj}(x) \; , \end{aligned} \qquad (2)$$

Extended Gai-Kapadia Model

1. GK 2010 zero recovery default mechanism.
2. Set of defaulted banks after n cascade steps: $\mathcal{D}^n \subset \mathcal{N}$.
3. Initial default probabilities: $p^{(0)}_{jk} = \mathbb{P}[v \in \mathcal{D}^0 | v \in \mathcal{N}_{jk}] = D_{jk}(0)$.
4. After n cascade steps:

$$\begin{aligned} \{v \in \mathcal{D}^n\} &= \{\bar{\Delta}_v \le \sum_{w \in \mathcal{N}^-_v} \bar{\Omega}_{wv} \mathbf{1}(w \in \mathcal{D}^{n-1})\} \\ p^{(n)}_{jk} &= \mathbb{P}[v \in \mathcal{D}^n | v \in \mathcal{N}_{jk}] \ . \end{aligned}$$

Zero Recovery Cascade Mapping Theorem

Theorem

Consider the LTI sequence of GK financial networks $(N, P, Q, \bar\Delta, \bar\Omega)$*. Then the following formulas hold with high probability as* $N \to \infty$*:*

1. *The quantities* $p^{(n)}_{jk}, \pi^{(n)}_k$ *satisfy the recursive formulas*

$$\begin{aligned} p^{(n)}_{jk} &= \langle D_{jk}, (\widetilde{w}^{(n-1)}_j)^{*j} \rangle \ , \\ \pi^{(n)}_k &= \sum_{j'} p^{(n)}_{j'k} P_{j'|k} \ , \\ \tilde{w}^{(n-1)}_j(x) &= \sum_{k'} Q_{k'|j} \Big((1-\pi^{(n-1)}_{k'})\delta_0(x) + \pi^{(n-1)}_{k'} w_{k'j}(x) \Big) \ . \end{aligned}$$

2. $\vec\pi^{(n)}$ *are a vector valued function* $G(\vec\pi^{(n-1)})$*, explicit in terms of RFN specification* $(N, P, Q, \bar\Delta, \bar\Omega)$*.*

Proving the Cascade Mapping Theorem

- This generalizes a well-known and very important result of Amini, Cont and Minca.
- Their proof is elegant and sophisticated. It relies on random graph technology known as Wormald's Theorem.
- Unfortunately, I'm not able to extend their proof to more complicated settings such as ours.
- We'd like to find a more flexible approach.

Core Strategy

- Work hard on an RFN with finite N.
- Determine the probability that any $v \in \mathcal{N}$ is in the set $\mathcal{D}^n$ conditionally on the finite skeleton.
- Notice that such probabilities are independent of N.
- Show that the expectation of this conditional probability can be expressed as a sum over configurations.
- Use ACG Theorem combined with the LTI property and Monotone Convergence to verify only tree configurations survive in the limit $N \to \infty$.

Proof of $N \to \infty$: a Template

1. Let $(\mathcal{N}, \mathcal{E}) \in \mathcal{G}^{(N)}$ be a fixed labelled configuration skeleton.
2. Use iterated conditioning to show:

$$\lim_{N\to\infty} \mathbb{E}[\mathbb{E}[\frac{1}{N} \sum_{v\in[N]} \mathbf{1}(v \in \mathcal{D}^n)|(\mathcal{N}, \mathcal{E})]] = \sum_{jk} p^n_{jk}$$

3. Insight: $\mathbb{E}[\mathbf{1}(v \in \mathcal{D}^n)|(\mathcal{N}, \mathcal{E})] = f^n(\mathcal{N}^{n-}_v)$ where $\mathcal{N}^{n-}_v$, a configuration, denotes the nth in-neighbourhood of v.
4. f^n is some complicated conditional expectation over $\{\bar{\Delta}, \bar{\Omega}\}$ that is independent of N!
5.

$$\lim_{N\to\infty} \sum_{\mathcal{N}^{n-}_v} \hat{p}^N(\mathcal{N}^{n-}_v) f^n(\mathcal{N}^{n-}_v) = \sum_{\mathcal{N}^{n-}_v} \lim_{N\to\infty} \hat{p}^N(\mathcal{N}^{n-}_v) f^n(\mathcal{N}^{n-}_v)$$

6. Verify using ACG Theorem and percolation logic that this equals desired formula.

GK: Mean Cascade Size and Cascade Frequency

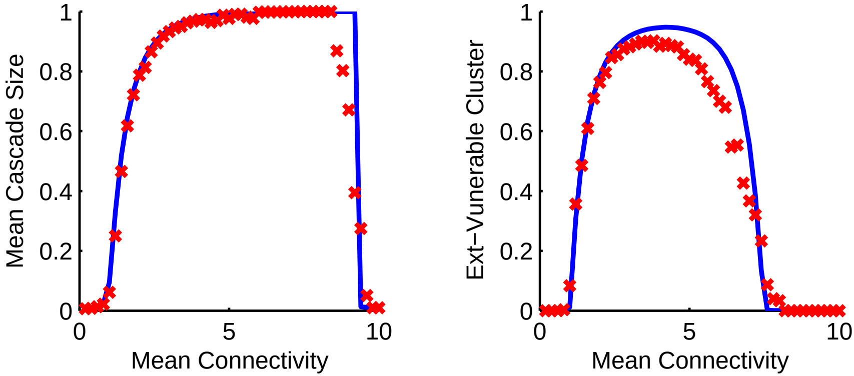


Figure: Mean fractional cascade size, and global cascade frequency as a function of mean degree z by large N analytics (blue curve) and by Monte Carlo simulation (red crosses).

GK: Assortative Networks

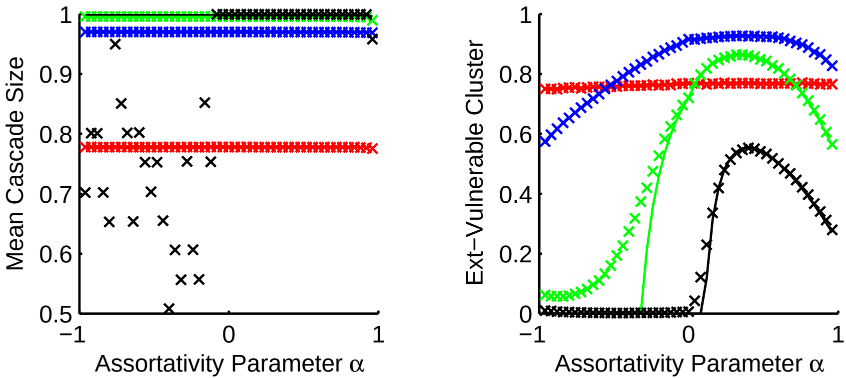


Figure: Mean cascade size and global cascade frequency as a function of positive and negative assortativity for four values of connectivity z: 1.5 (red), 3.5 (blue), 5.5 (green), 7.5 (black).

GK: Exposure Uncertainty

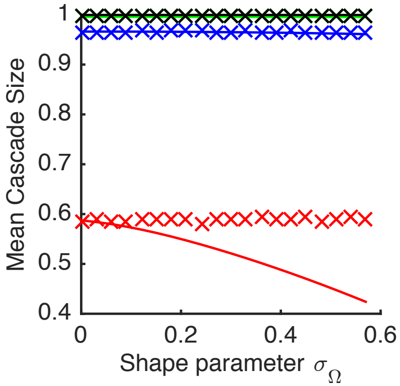


Figure: Mean cascade size and global cascade frequency vs exposure standard deviation σ_Ω for four values of connectivity z : 1.5 (red), 3.5 (blue), 5.5 (green), 7.5 (black).

Main Contributions of the Book

1. Places financial cascade mechanisms on common mathematical footing.
2. Develops a rich framework of RFN specifications.
3. Provides exact large N analytic formulas for cascades.
4. Introduces alternative to Amini-Cont-Minca's method of proof for $N \to \infty$.
5. Provides numerical validation of the cascade mapping theorem.

Future Directions

1. Develop LTI algorithms for a variety of full and partial recovery cascade mechanisms, such as EN 2001, liquidity hoarding, fire sale models and their extensions.
2. Investigate limits of the LTI approximation.
3. Abstract aspects of Financial Systemic Risk.
4. Explore RFN cascade models with IRG skeletons with community and multiplex structure.
5. Calibrate joint distributions of balance sheets and exposures from systemic risk databases.
6. Justify bank crisis behaviour assumptions using the theory of global games (see Morris and Shin).

Thanks

…to many friends, colleagues and collaborators for sharing ideas and insight into an exciting field.